

Summary for 9709 Pure Mathematics 2 & 3

Algebra

Modulus

The modulus function $|x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Equations

If you are given an equation in the form $|ax + b| = k$, you can solve the equation using $ax + b = k$ and $ax + b = -k$.

Graphs

We draw the graph of $y = |f(x)|$ by mirroring everything below the x -axis on a $y = f(x)$ graph about the x -axis.

Inequalities

Remember these properties:

$$|a| \leq b \Leftrightarrow -b \leq a \leq b$$
$$|a| \geq b \Leftrightarrow a \leq -b \text{ or } a \geq b$$

When solving an inequality involving a quadratic with roots α and β is greater than or equal to zero, where $\alpha \leq \beta$, $x \leq \alpha$ and $x \geq \beta$. When the quadratic is lesser than or equal to zero, the opposite is true.

Division of polynomials

Polynomials can be divided with long division.

$P(x) = d(x) \cdot q(x) + r$ where $P(x)$ is the dividend, $d(x)$ is the divisor, $q(x)$ is the quotient, and r is the remainder. This is called the **division algorithm for polynomials**.

The factor theorem and the remainder theorem

For a polynomial $P(x)$, if $P(c) = 0$ then $(x - c)$ is a factor of $P(x)$; if $P(\frac{b}{a}) = 0$ then $(ax - b)$ is a factor of $P(x)$.

When a polynomial $P(x)$ is divided by $(x - c)$, the remainder is $P(c)$; when divided by $(ax - b)$, the remainder is $P(\frac{b}{a})$.

Improper algebraic fractions

An algebraic fraction is said to be improper if the degree of the polynomial on the numerator is higher than that of the denominator. In the opposite case, the fraction is proper.

Partial fractions

$$\frac{px + q}{(a_1x + b_1)(a_2x + b_2)\dots(a_nx + b_n)} \equiv \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n}$$

where n is any positive integer greater than or equal to 2 and A_1 to A_n are to be found.

$$\frac{px + q}{(ax + b)^2} \equiv \frac{A}{ax + b} + \frac{B}{(ax + b)^2}$$

$$\frac{px + q}{(ax + b)(cx^2 + d)} \equiv \frac{A}{ax + b} + \frac{Bx + C}{cx^2 + d}$$

Binomial expansions

$$(a + x)^n = a^n \left[1 + nx + \frac{n(n-1)}{2!} \left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{x}{a}\right)^3 + \dots \right] \text{ for } \left|\frac{x}{a}\right| < 1$$

Trigonometry

$$a \sin(\theta) \pm b \cos(\theta) = R \sin(\theta \pm \alpha)$$

$$a \cos(\theta) \pm b \sin(\theta) = R \cos(\theta \mp \alpha)$$

where $R = \sqrt{a^2 + b^2}$ and $\tan(\alpha) = \frac{b}{a}$

All other necessary identities are given in the formula booklet.

Calculus

Anti-derivatives and derivatives of needed functions are given in the formula booklet.

The trapezium rule

The approximate value of $\int_a^b f(x) dx$ can be found by considering that the region under the curve can be thought of as many very tiny trapeziums. We can divide the curve into n strips each with width h to calculate the approximate value of the integral as:

$$\int_a^b f(x) dx \approx \frac{h}{2} [y_0 + y_n + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})]$$

where $h = \frac{b-a}{n}$

Integration by substitution

When a function of x is made up of other functions of x , we could substitute a variable, say u , for one of the constituent functions, find dx in terms of du , substitute the expression involving du in place of dx , simplify the expression being integrated, then integrate with respect to u .

For example, consider integrating the function $\cos(x^2)2x$. Let $u = x^2$.

$$du = 2x dx \Rightarrow dx = \frac{du}{2x}$$

$$\int \cos(x^2)2x dx = \int \cos(u)2x \frac{du}{2x} = \int \cos(u) du = \sin(u) + C = \sin(x^2) + C$$

Integration by parts

If we want to find the integral of $f(x)$ where we can write $f(x)$ as $f(x) = u \frac{dv}{dx}$,

$$\int f(x) dx = \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

This is given in the formula booklet.

Integrating rational functions

When rational functions are split into partial functions, we often see the form $\frac{kf'(x)}{f(x)}$. All of the splitting of rational functions above are in this form (an easy rule to remember if you forget how to derive partial fractions during the test).

$$\int \frac{kf'(x)}{f(x)} dx = k \ln(|f(x)|) + c$$

Differential equations

Differential equations in this syllabus can be solved by integrating both sides of the equation once rearranged such that the equation is of the form $f(y)\frac{dy}{dx} = g(x)$.

$$f(y)\frac{dy}{dx} = g(x) \Rightarrow \int f(y)\frac{dy}{dx} dx = \int g(x) dx \Rightarrow \int f(y) dy = \int g(x) dx$$

The general solution to a differential equation doesn't consider example points on the graph to find the constants of integration. The particular solution does.

Numerical solutions of equations

For $f(x) = 0$, if the sign of $f(\alpha)$ and $f(\beta)$ are not the same, there lies a root between α and β .

We use this as our starting point for what's called the iterative process. There are other methods to find numerical solutions, but we only need the iterative process for the syllabus.

Click here for a clearer explanation.

We first rearrange $f(x) = 0$ to the form $x = F(x)$. When α is a value of x such that $x = F(x)$ to the required precision, then α is a solution of $f(x) = 0$.

We use our formula with x as the subject to create a formula where each iteration is based on the result of the previous iteration, i.e. $x_{n+1} = f(x_n)$. We list the values of x for different iterations until the current iteration has the same value as the previous (to the required precision) at which point we say that that value is a solution to $f(x) = 0$.

Vectors

Scalar product / dot product

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 \cdot z_2$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta \text{ where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}$$

Vector equation of a line

A line's vector equation is defined by the resultant vector formed by adding a position vector to a scalar multiple of a direction vector.

$$\vec{r} = \vec{a} + \lambda \vec{d}$$

\vec{a} can be the position vector of any point on the line.

Cartesian equations (parametric equations) can be formed by considering the components of the vectors individually. They can be used to find the point of intersection of the lines.

Planes

Equation type	Form
Scalar	$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$
Cartesian	$ax + by + cz = d$
Vector	$\vec{r} = \vec{a} + \vec{b}s + \vec{c}t$

The angle between two planes can be found by comparing the angle between their normals. The same can be said of the angle between a plane and a line.

To find the line of intersection between two planes, we set one of the variables as a free variable, then obtain a set of equations from the vector equation.

To show that a line lies on a plane, substitute the parametric equations of the line into the Cartesian equation of the plane.

For a line meeting the plane, substitute its parametric equations into the Cartesian equation of the plane to find at which point they intersect.

For the shortest distance between the point P and the plane $\vec{r} \cdot \vec{n} = d$ use $\left| \frac{PQ \cdot \vec{n}}{|\vec{n}|} \right|$, where Q is any chosen point on the plane.

Complex numbers

$$i = \sqrt{-1} \Rightarrow i^2 = -1$$

Complex numbers, unlike real numbers that are represented on a number line, are represented in two dimensions, one dimension for the real part of the number and another for the imaginary part. They can be written in the Cartesian form as $x + iy$

The complex conjugate of $z = x + iy$ is defined as $z^* = x - iy$

The x and y can act as the corresponding Cartesian coordinates when plotting complex numbers on an Argand diagram.

A complex number can also be represented in the modulus-argument form (similar to representing points on a polar coordinate system).

The **modulus** of a complex number $z = x + iy$, $|z|$ is the magnitude of vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

The **argument**, $\arg z$, is the direction of $\begin{pmatrix} x \\ y \end{pmatrix}$. It is the angle between the positive real axis and the position vector.

The **principal argument**, θ , of a complex number is an angle such that $-\pi < \theta \leq \pi$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \arg z = \arctan\left(\frac{y}{x}\right)$$

The **modulus-argument form** of our complex number is

$$r(\cos \theta + i \sin \theta)$$

The **exponential form** is

$$e^{i\theta}$$

$$|z_1 z_2| = |z_1| |z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

You can divide two complex numbers by multiplying both the numerator and the denominator by the complex conjugate of the denominator.

If a polynomial has a complex number as its root, the complex conjugate of that complex number is also a root, so complex roots appear in pairs.

Cube roots of unity

$$z^3 = 1$$

$$z = 1$$

So $(z - 1)$ is a factor of $z^3 - 1$.

$$z^3 - 1 = 0$$

$$z^3 - 1 = (z - 1)(z^2 + z + 1)$$

$$(z - 1)(z^2 + z + 1) = 0$$

$$(z^2 + z + 1) = 0$$

$$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Loci

Loci can be drawn and regions shaded on an Argand diagram, similar to how it is done on a Cartesian plane.

Circles

Circles are defined by their centre and radius.

- $|z| = r$ is the circumference of a circle with origin as centre and radius r .
- $|z| < r$ is all points within a circle with origin as centre and radius r .
- $|z| \leq r$ is all points within and the circumference of a circle with origin as centre and radius r .

Replace z with $z - z_1$ and the coordinates of z_1 becomes the center of the circle.

Lines

Lines are defined by their starting point and their angle from the horizontal.

- $\arg(z - z_1) = \alpha$ is a line that starts at the coordinates of z_1 and is α clockwise from the horizontal.
- $\arg(z - z_1) < \alpha$ represents all the points on all the lines from 0 radians from the horizontal to α clockwise from the horizontal, starting at the coordinates of z_1 and is α from the horizontal.

Multiply the expression inside $\arg()$ with -1 and α is now the counter-clockwise angle from the horizontal.

Perpendicular bisector

$|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line from z_1 to z_2 . The line equation of the locus formed by the equation can therefore be found by calculating the gradient and intercept like we would on a Cartesian plane.