Summary for 9213 Further Mechanics

Projectiles

Launched upwards

The initial velocity of a projectile in 2 dimensions can be described by its two components: $u_x = u\cos\theta$ and $u_y = u\sin\theta$.

As the projectile motion continues however, gravity acts on the projectile which changes v_y, while no force acts in the horizontal direction, allowing v_x to remain constant.

$$v_y = u_y + a_y t \Rightarrow v_y = u\sin\theta - gt$$
$$v_x = u\cos\theta$$

The displacement of any particle at a given time is given by

$$s = ut + \frac{1}{2}at^2$$

From this we form equations for y and x (displacements in the vertical and horizontal directions respectively).

$$y = u_y t + \frac{1}{2}a_y t^2 \Rightarrow y = ut\sin\theta - \frac{1}{2}gt^2$$
$$x = ut\cos\theta$$

At the highest point, $v_y = 0$ (otherwise the particle would keep moving upwards), and this covers half the time of the motion. If the total time for the motion is t, with the first half taking time t_1 and the second half t_2, then

$$u\sin\theta = gt_1 \Rightarrow t_1 = \frac{u\sin\theta}{g}$$
$$t_2 = t_1$$
$$t = t_1 + t_2$$
$$t = 2t_1$$
$$t = \frac{2u\sin\theta}{g}$$

Projection from a raised platform

When a particle is projected upwards from a raised platform, any position below the starting point is a negative displacement.

You can calculate, for example, the final velocity of a projectile projected upwards from a raised platform applying $v_y^2 = u_y^2 + 2as$ to the vertical velocity, where s becomes the negative of the height of the platform (as the displacement of the particle is from the top of the platform to the ground), and $a = -g$. Then consider that the horizontal velocity remains constant throughout a projectile motion, and finally use $v = \sqrt{v_y^2 + v_x^2}$.

Launched downwards

When launced downwards, the gravitational acceleration is positive in the equation for the vertical velocity, as we consider positive any downward velocity.

The Cartesian equation of the trajectory

$$x = ut\cos\theta \Rightarrow t = \frac{x}{u\cos\theta}$$

$$y = ut\sin\theta - \frac{1}{2}gt^2$$

$$y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\therefore y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta$$

This is a quadratic equation that forms an inverted parabola.

Equilibrium of a rigid body

An object is in a state of equilibrium when all forces acting on it are balanced and the vector sum of the forces is zero.

The moment of a force about a point is the product of the force and the perpendicular distance from the point to the line of action of the force.If the distance is zero, there is no moment so there is no turning effect.

A negative clockwise turning effect is a positive anticlockwise turning effect, and vice versa.

Centres of mass of rods and laminas

All the weight of an object acts through its centre of mass.

The distance of the centre of mass $\bar{x}$ from the point of reference:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

We can extend this to all of the axes in a coordinate system.

When working with 2D frameworks in this syllabus, consider gravity as acting towards the paper on which it is printed.

Laminas

Laminas are 2 dimensional shapes (frameworks are formed with lines). If a lamina is uniform, the mass is spread evenly across its area.

The centre of mass of a triangular lamina is given by:

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

The centre of mass of a sector-shaped lamina with an angle of 2α radians subtended at the centre, in **polar coordinates**:

$$\left(\frac{2r\sin\alpha}{3\alpha}, \frac{\alpha}{2}\right)$$

The centre of mass between two objects is the barycentre.

Shapes formed from wire

A piece of uniform wire, with density ρ could be bent into an arc with radius r, and angle 2α radians subtended at the centre.

The total mass of the wire is therefore $2r\rho\alpha$, so the moment about Oy is:

$$2r\rho\alpha\bar{x} = \int_{-\alpha}^{\alpha} r^2 \rho \cos\theta \,\mathrm{d}\theta$$

$$2\alpha\bar{x} = r\int_{-\alpha}^{\alpha} \cos\theta \,\mathrm{d}\theta$$

$$\therefore \bar{x} = \frac{r\sin\alpha}{\alpha}$$

Centres of mass of solids

Cone

For a right circular cone, with its height parallel to the x-axis:

$$\bar{x} = \frac{3}{4}h$$

The y-coordinate is 0, and the z-coordinate is that of the centre of the base circle.

Hemisphere

The centre of mass is $\frac{3}{8}r$ from the centre of the plane face.

Combining

When combining solids, add the centres of mass from an axis to find the total moment, find the total mass and use $m\bar{x} = M$ where m is the total mass and M is the total moment.

Circular Motion

Horizontal circles

$$v = r\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow v = r\omega$$

- ω is the angular speed, measured in radians per second.
- v is the velocity

A complete revolution is 2π radians, so the frequency (revolutions per second) is $\frac{2\pi}{\omega}$.

The acceleration towards the center for an object in circular motion is

$$a = \frac{v^2}{r} \Rightarrow a = r\omega^2$$

The 3D case in horizontal circles

Conical pendulum

Consider a particle on the end of a light, inelastic string with the other end attached to a fixed point, as the particle travels in a circular path. This is a conincal pendulum model.

$$R(\uparrow): T\cos\theta = mg$$
$$R(\leftarrow): T\sin\theta = ma = mr\omega^2$$
$$\sin\theta = \frac{r}{l} \Rightarrow T\sin\theta = T \times \frac{r}{l}$$
$$T\sin\theta = mr\omega^2 \Rightarrow T = ml\omega^2$$

We can derive an equation for the angular speed from the above.

$$\omega = \sqrt{\frac{g}{l\cos\theta}}$$

Objects on a sloped surface

On a sloped surface we have to account for friction and gravity.